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Abstract: In this paper, we propose a new robust receding horizon control scheme for linear input-constrained discrete-time systems with polytopic uncertainty. We provide a rigorous proof for closed-loop stability. The control scheme is based on the minimization of the worst-case cost with time-varying terminal weighting matrices, which can easily be implemented by using linear matrix inequality optimization. We discuss modifications of the proposed scheme that improves feasibility or on-line computation time. We compare the proposed schemes with existing results through simulation examples.

Key-words: RHC, constrained, robust control, polytopic uncertainty, stability, feasibility, computation time.

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Conception de contrôles robustes à horizon fuyant pour des systèmes contraints et polytopiques-incertains

Résumé : Nous proposons dans cet article un nouveau cadre pour le contrôle robuste à horizon fuyant pour des systèmes linéaires, discrets en temps et contraints en entrée et polytopiques-incertains. Nous donnons une preuve rigoureuse pour la stabilité fermée. Ce cadre est basé sur la minimisation du coût le pire avec des matrices variables dans le temps de poids associées à l'état terminal qui peuvent facilement être obtenues par des optimisations basées sur des inégalités matricielles. Nous discutons des modifications du cadre proposé qui améliorent la faisabilité et le calcul en ligne. Nous comparons ce travail rapport aux cadres existants par la simulation de divers exemples.

Mots-clés : RHC, contraintes, contrôle robuste, incertitude polytopique, stabilité, faisabilité, temps du calcul.

1 Introduction

Receding horizon control (RHC) is a closed-loop control strategy based on optimization and can consider a finite-horizon cost function. These two properties enable the RHC to handle input/state constraints, uncertain systems, nonlinear systems, time-varying tracking command, disturbances, etc. For this reason, the RHC has been widely investigated as a successful feedback strategy [1, 2, 3, 4, 5].

How to design a robust control for uncertain systems is an important issue in the control literature. The robust RHC problem is much more complicated even for linear systems in the presence of input/state constrains since then we need to consider feasibility as well as performance and on-line computational burden on a controller. This issue has been addressed for stable single-input-single output (SISO) systems based on a L_1 cost function [6, 7]. For general open-loop linear systems, the paper [1] addresses this issue based on the quadratic cost function and implements the on-line optimization problem via the linear matrix inequality (LMI) technique.

However, the proposed method in [1] uses a fixed linear state-feedback control scheme every instant which is easy to implement and calculate, but is very conservative in terms of performance and feasibility for constrained systems. Note that RHC with the linear state-feedback scheme is not fully optimal even for linear systems since the closed-loop systems are nonlinear in the presence of constraints. Therefore, it has a very small feasible initial-state set and does not perform well. In order to overcome these shortcomings, nonlinear feedback RHCs have been suggested [8, 9]. However, since the paper [8] allows only one horizon, it still has a small feasible initial-state set and a performance limit. The nonlinear feedback RHC in [9] has a large feasible set and a better performance than those in [1, 8] by allowing more than one horizon. However, it has a lot of computational burden due to many design variables, and seems to be difficult to understand and implement. In addition, the proofs of closed-loop stability in [8, 9] miss some details as mentioned later in this paper. For time-varying systems, we should be very careful for stability analysis and implementation as shown in [4]. Thus, it will be interesting to develop a simple stabilizing RHC scheme that has a better performance, less computational burden or a large feasible set for constrained systems with polytopic uncertainty.

In this paper, we propose a new robust RHC scheme for linear input-constrained discrete-time systems with polytopic uncertainty that can easily be implemented via LMI optimization. The control scheme is based on the minimization of the worst-case finite horizon cost with time-varying terminal weighting matrices. We provide a rigorous proof of the closed-loop stability of the proposed scheme. We discuss modifications of the proposed scheme that improve feasibility or on-line computation time. Through simulation examples, we show that the proposed schemes have a better performance, a less computational burden, or a wider feasible set than existing results.

In Section 2, we propose a new robust RHC scheme and prove its stability rigorously. In Section 3, we implement the proposed schemes via the LMI optimization. In Section 4, we illustrate our results through simulation examples. Finally, we present conclusions in Section 5.

2 Robust Receding Horizon Control

Consider a linear discrete time-varying system with polytopic uncertainty:

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i), \quad x(0) = x_0 \\ (A(i), B(i)) &\in \Omega \end{aligned} \tag{1}$$

subject to input constraints

$$u_{\text{min}} \le G_u u(i) \le u_{\text{max}}, \quad i = 0,\ 1,\ \cdots,\ \infty, \tag{2}$$

where $x(i) \in R^n$ is the state, $u(i) \in R^m$ the control, $G_u(i) \in R^{l\times m}$, $u_{\text{min}} < 0$, and $u_{\text{max}} > 0$. Ω is the polytopic set with L-vertices:

$$\Omega = \{(A(i), B(i)) \mid (A(i), B(i)) = \lambda_1(i)(A_1, B_1) + \lambda_2(i)(A_2, B_2) + \cdots + \lambda_L(i)(A_L, B_L)\},$$

where $\sum_{j=1}^{L} \lambda_j(i) = 1$ and $\lambda_j(i) \ge 0$. In [1], [8], [9], most of existing RHC schemes assume that $u_{\text{min}} = -u_{\text{lim}}$ and $u_{\text{max}} = u_{\text{lim}}$ for some positive constant vector u_{lim}.

Throughout the rest of this paper, we assume that the pair $(A(i), B(i)) \in \Omega$ is unknown for all times and uniformly stabilizable, and the pair $(A(i), C(i))$ is uniformly observable. For this system, consider the following optimization problem:

$$\begin{aligned} J^*(i, i+N) = & \underset{u(\cdot|i),\ Q_f(i)}{\text{Minimize}} \ \underset{(A(\cdot|i), B(\cdot|i))\in\Omega}{\text{Maximize}} \sum_{\tau=i}^{i+N-1} [x^T(\tau|i)Qx(\tau|i) + u^T(\tau|i)Ru(\tau|i)] \\ & + x^T(i+N|i)Q_f(i)x(i+N|i) \end{aligned} \tag{3}$$

subject to input constraints

$$u_{\text{min}} \le G_u u(\tau|i) \le u_{\text{max}}, \quad \tau = i,\ i+1,\ \cdots,\ i+N-1, \tag{4}$$

where $Q = C^T C \ge 0$, $R = R^T > 0$, and $Q_f(i) = Q_f^T(i)$ is positive definite. The terminal weighting matrix $Q_f(i)$ is a key design parameter for feasibility and closed-loop stability. We study later in this paper how to design $Q_f(i)$.

Since (3) subject to (4) is difficult to solve directly, we investigate an equivalent optimization problem that can be solved via LMI optimization. For this purpose, we introduce the following two lemmas.

Define $\Psi_j(i)$ and $\Psi_b(i)$ as

$$\begin{aligned} \Psi_j(i) &= (A_j(i)x(i) + B_j(i)u(i))^T \Psi(i)(A_j(i)x(i) + B_j(i)u(i)), \quad j \in [1, L] \\ \Psi_b(i) &= (A(i)x(i) + B(i)u(i))^T \Psi(i)(A(i)x(i) + B(i)u(i)), \end{aligned}$$

where $(A_j(i), B_j(i)) \in \Omega_*$ and $\Omega_* = \{(A_1, B_1),\ (A_2, B_2),\ \cdots,\ (A_L, B_L)\}$.

Lemma 1 *For any $\Psi(i) \geq 0$, $(A(i), B(i)) \in \Omega$, $x(i)$, $u(i)$, G_1 and G_2, we have*

$$\max_{j\in[1,L]} \Psi_j(i) \quad \geq \quad \Psi_b(i) \;\; \textit{and} \tag{5}$$

$$\begin{aligned}&\max_{j\in[1,L]} [G_1(A_jx(i) + B_ju(i)) + G_2]^T\Psi(i)[G_1(A_jx(i) + B_ju(i)) + G_2] \;\geq \\ &\quad [G_1(A(i)x(i) + B(i)u(i)) + G_2]^T\Psi(i)[G_1(A(i)x(i) + B(i)u(i)) + G_2].\end{aligned} \tag{6}$$

Proof: See Appendix A. ■

The relation (5) is a simple extension of the result in [1] with $u(\tau|i) = -H(i)x(\tau|i)$ for all $\tau \geq i$. The subsequent lemma extends Lemma 1 to a finite horizon case.

Lemma 2 *For any $\Psi(\tau|i) \geq 0$, $(A(\tau|i), B(\tau|i)) \in \Omega$, $x(i)$, $u(\tau|i)$ and $T \geq i$, we have*

$$\max_{(A_j(\cdot|i),B_j(\cdot|i))\in\Omega_*} \sum_{\tau=i}^{T} \Psi_j(\tau|i) \;\geq\; \sum_{\tau=i}^{T} \Psi_b(\tau|i), \tag{7}$$

where the state trajectories of $\Psi_j(\tau|i)$ and $\Psi_b(\tau|i)$ are $x^1(\tau+1|i) = A_j(\tau|i)x^1(\tau|i)+B_j(\tau|i)u(\tau|i)$ and $x^2(\tau+1|i) = A(\tau|i)x^2(\tau|i) + B(\tau|i)u(\tau|i)$, respectively with the same initial state $x^1(i|i) = x^2(i|i) = x(i)$.

Proof: See Appendix B. ■

Lemma 2 enables us to convert the problem (3) to the equivalent one

$$\underset{U(i),\ Q_f(i)}{\text{Minimize}} \quad \gamma(i), \tag{8}$$

where

$$\begin{aligned}\gamma(i) \quad \geq \quad & \max_{(A_j(\cdot|i),B_j(\cdot|i))\in\Omega_*} \sum_{\tau=i}^{i+N-1} [x^T(\tau|i)Qx(\tau|i) + u^T(\tau|i)Ru(\tau|i)] \\ & + x^T(i+N|i)Q_f(i)x(i+N|i).\end{aligned} \tag{9}$$

The above optimization problem can also be converted to the equivalent one, which can be much more flexible in implementation than (8) subject to (9) as shown in this paper later,

$$\underset{U(i),\ Q_f(i)}{\text{Minimize}} \quad \gamma_1(i) + \gamma_2(i), \tag{10}$$

where

$$\gamma_1(i) \quad \geq \quad \max_{(A_j(\cdot|i),B_j(\cdot|i))\in\Omega_*} \sum_{\tau=i}^{i+N-1} [x^T(\tau|i)Qx(\tau|i) + u^T(\tau|i)Ru(\tau|i)] \tag{11}$$

$$\gamma_2(i) \quad \geq \quad x^T(i+N|i)Q_f(i)x(i+N|i). \tag{12}$$

The resulting solutions at each time i are denoted as $u^*(\tau|i)$ and $(A_j^*(\tau|i), B_j^*(\tau|i))$ for $\tau \in [i, i+N-1]$. The first control $u^*(i)(= u^*(i|i))$ is called the receding horizon control (RHC).

Note that the above optimization problems (8) and (10) have less design variables than [9], while the problem (10) is the same as those in [8], [9] when $N = 1$.

For the problems (8) and (10), we have only to consider L^N number of sets Ω_N, where

$$\Omega_k = \{(A_j(i|i), B_j(i|i)),\ (A_j(i+1|i), B_j(i+1|i)),\ \cdots,\ (A_j(i+k-1|i), B_j(i+k-1|i))\}. \quad (13)$$

Next, in order to investigate feasibility and stability of the proposed RHC scheme, we suggest a lemma, which is a simple extension of the result in [10] for unconstrained continuous time-invariant systems.

Lemma 3 *For any $(A(\cdot|i), B(\cdot|i)) \in \Omega$, $Q_f(i) \geq 0$, $x(i)$ and $U(i)$, we have*

$$\max_{(A_j(\cdot|i),B_j(\cdot|i))\in\Omega_*} x_a^T(i+N|i)\, Q_f(i)\, x_a(i+N|i) \quad \geq \quad x_b^T(i+N|i)\, Q_f(i)\, x_b(i+N|i), \quad (14)$$

where

$$\begin{aligned}
U(i) &= [u^T(i|i),\ u^T(i+1|i),\ \cdots,\ u^T(i+N-1|i)]^T \\
x_a(i+N|i) &= \Phi_{i+N,i}(A_j)x(i) + B_{\phi_j}(i)U(i),\ \ x_b(i+N|i) \ = \ \Phi_{i+N,i}(A)x(i) + B_\phi(i)U(i) \\
\Phi_{\sigma,\tau}(A_j) &= A_j(\sigma-1|\tau)A_j(\sigma-2|\tau)\cdots A_j(\tau|\tau) \\
\Phi_{\sigma,\tau}(A) &= A(\sigma-1|\tau)A(\sigma-2|\tau)\cdots A(\tau|\tau) \\
B_{\phi_j}(i) &= [\Phi_{i+N,i+1}(A_j)B_j(i|i),\ \ \Phi_{i+N,i+2}(A_j)B_j(i+1|i),\ \ \Phi_{i+N,i+3}(A_j)B_j(i+2|i), \\
&\quad \cdots,\ \ B_j(i+N-1|i)] \\
B_\phi(i) &= [\Phi_{i+N,i+1}(A)B(i|i),\ \ \Phi_{i+N,i+2}(A)B(i+1|i),\ \ \Phi_{i+N,i+3}(A)B(i+2|i), \\
&\quad \cdots,\ \ B(i+N-1|i)].
\end{aligned}$$

Proof: The proof here follows that of Lemma 2. ■

Now, we are ready to investigate feasibility of (8) (or (10)) subject to (4).

Lemma 4 *Assume that there exists $Q_f(i)$ satisfying*

$$Q_f(i) \quad \geq \quad \max_{j\in[1,L]} (A_j - B_jH(i))^T Q_f(i)(A_j - B_jH(i)) \ \textit{ for some } \ H(i) \quad (15)$$

$$\begin{bmatrix} E_u(i) & G_uY(i) \\ Y^T(i)G_u^T & S(i) \end{bmatrix} \quad \geq \quad 0, \quad E_{u,(j,j)}(i) \ \leq \ u_{lim,j}^2, \quad (16)$$

where $S(i) = \gamma_2(i)Q_f^{-1}(i)$, $Y(i) = H(i)S(i)$, $u_{lim,j}$ and $E_{u,(j,j)}(i)$ are the jth element of u_{lim} and the (j,j) element of $E_u(i)$, respectively.

If (8) (or (10)) subject to (4), (15), (16) and $x_a^*(i+N|i) \in \mathcal{E}_{Q_f(i)}$ is feasible at the initial time $i = 0$, it is always feasible, where $x_a^*(i+N|i)$ is the solution of the left side in (14) and $\mathcal{E}_{Q_f(i)}$ is an ellipsoid defined as

$$\mathcal{E}_{Q_f(i)} \quad = \quad \{\xi \mid \xi^T Q_f(i)\xi \ \leq \ \gamma_2(i)\}. \quad (17)$$

Proof: We have only to show that feasible solutions at i are feasible ones at $i+1$ for any i. With $\sigma = i+N$, let $x(\sigma|i+1)$ be the state at σ due to the initial state $x(i+1|i+1)(= A(i)x(i)+B(i)u^*(i|i))$, $u(\tau|i+1) = u^*(\tau|i)$, and arbitrary $(A_j(\tau|i+1), B_j(\tau|i+1)) \in \Omega_*$ for $\tau \in [i+1, \sigma-1]$. Then, $x(\sigma|i+1)(= x_b(\sigma|i)) \in \mathcal{E}_{Q_f(i)}$ from Lemma 3. Thus, $u(\sigma|i+1) = -H(i)x(\sigma|i+1)$ satisfies the input constraints (4) from (16) as shown in [10]. Hence, we can have feasible solutions at $i+1$ such as $u(\tau|i+1) = u^*(\tau|i)$ for $\tau \in [i+1, \sigma-1]$ and $u(\sigma|i+1) = -H(i)x(\sigma|i+1)$. Next, we show that we can have $x_a^*(\sigma+1|i+1) \in \mathcal{E}_{Q_f(i+1)}$. With $Q_f(i+1) = Q_f(i)$, for all vertices (A_j, B_j), $x^T(\sigma+1|i+1)Q_f(i)x(\sigma+1|i+1) = x^T(\sigma|i+1)(A_j - B_jH(i))^T\ Q_f(i)$ $(A_j - B_jH(i))x(\sigma|i+1) \leq x^T(\sigma|i+1)Q_f(i)x(\sigma|i+1)$ by (15). Since $x(\sigma|i+1)(= x_b(\sigma|i)) \in \mathcal{E}_{Q_f(i)}$, we can have $x_a^*(\sigma+1|i+1) \in \mathcal{E}_{Q_f(i+1)}$. ■

Next, we investigate closed-loop stability of RHC by using the following additional lemma.

Lemma 5 *If $Q_f(i)$ satisfies (12), (16), and*

$$Q_f(i) \quad \geq \quad Q + H^T(i)RH(i) + \max_{j\in[1,L]}(A_j - B_jH(i))^TQ_f(i)(A_j - B_jH(i)) \text{ for some } H(i), \tag{18}$$

then $J^*(i, i+N) \geq x^T(i)Qx(i) + u^{*T}(i)Ru^*(i) + J^*(i+1, i+N+1)$, where $x(i+1) = A(i)x(i)$ $+ B(i)\, u^*(i)$.

Proof: See Appendix C. ■

Theorem 1 *Assume that problem (10) subject to (4), (11), (12), (16), and (18) (or (8) subject to (4), (9), (16), (18), and $x_a^*(i+N|i) \in \mathcal{E}_{Q_f(i)}$ with $\gamma_2(i) = 1$) is feasible at the initial time. Then the closed-loop system with the resulting RHC is uniformly asymptotically stable.*

Proof: Lemma 4 shows that the optimization problem is always feasible. Since N is finite and the input is constrained, $J^*(0, N)$ is finite and $J^*(i, i+N)$ is thus bounded. Lemma 5 shows that $u^*(i) \to 0$ as $i \to \infty$ and $x(i) \to 0$ as $i \to \infty$ independently of i since the system is uniformly observable. Since the closed-loop system is uniformly attractive and becomes the unconstrained system near the equilibrium, it is uniformly asymptotically stable (see [3] for the detailed technique). ■

The results in [8] and [9] prove the asymptotic stability under the uniform detectability or $Q \geq 0$. However, they just proved the attractivity by showing $x(i) \to 0$ as $i \to \infty$. The attractivity does not mean the asymptotic stability directly.

The following remark compares the proposed RHC with the existing ones in terms of feasibility and on-line computational burden.

Remark 1 *The proposed RHC from (10) subject to (4), (11), (12), (16), and (18) has the computation time smaller than that in [9], while they have the same feasible initial-state set. Later in this paper, simulation examples show that they have the similar performance. Compared with the RHC from (10) and the result in [9], the RHC from (8) subject to (4), (9), (16), (18), and $x_a^*(i+N|i)$ $\in \mathcal{E}_{Q_f(i)}$ with $\gamma_2(i) = 1$ has a smaller computation time, while it has a smaller feasible set due to $x_a^*(i+N|i) \in \mathcal{E}_{Q_f(i)}$ with $\gamma_2(i) = 1$.*

As a way to reduce computational burden of the proposed stabilizing RHC scheme in Theorem 1, we suggest the following scheme:

Corollary 1 *Let $Q_f(0)$, $H(0)$, and $\gamma_2(0)$ be the solutions for (16) and (18) at the initial time. Assume that problem (8) subject to (4), (9), and $x_a^*(i+N|i) \in \mathcal{E}_{Q_f(0)}$ with $Q_f(i)$, $H(i)$, and $\gamma_2(i)$ replaced by $Q_f(0)$, $H(0)$, and $\gamma_2(0)$ is feasible at the initial time. Then the closed-loop system with the resulting RHC $u^*(i)$ is uniformly asymptotically stable.*

In the next section, we introduce how to implement the proposed schemes.

3 Implementation of the proposed equations via LMI

For simple presentation of matrices, define $\bar{F}_j(i)$ and $\hat{F}$ as

$$\bar{F}_j(i) \;=\; \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ F_j(i|i) & \ddots & \ddots & \ddots & \vdots \\ 0 & F_j(i+1|i) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & F_j(i+N-2|i) & 0 \end{bmatrix}, \text{ and } \hat{F} \;=\; \begin{bmatrix} F & 0 & \cdots & 0 \\ 0 & F & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & F \end{bmatrix}.$$

$\bar{F}_j(i)$ and $\hat{F}$ are used for presentation of $\bar{A}_j(i)$, $\bar{B}_j(i)$, $\hat{Q}$, $\hat{R}$, and $\hat{G}_u$, which consist of $A_j(\cdot|i)$, $B_j(\cdot|i)$, Q, R, and G_u, respectively.

Then, we can convert (4), (11), (12), and (18) to the equivalent forms

$$[u_{\min}^T, \cdots, u_{\min}^T]^T \;\le\; \hat{G}_u U(i) \;\le\; [u_{\max}^T, \cdots, u_{\max}^T]^T \tag{19}$$

$$\begin{bmatrix} \gamma_1(i) - V_1(i)U(i) - V_0(i) & (V_2^{\frac{1}{2}}(i)U(i))^T \\ V_2^{\frac{1}{2}}(i)U(i) & I \end{bmatrix} \;\ge\; 0 \tag{20}$$

for all L^{N-1} number of sets Ω_{N-1} in (13),

$$\begin{bmatrix} 1 & (\Phi_{i+N,i}(A_j)x(i) + B_{\phi_j}(i)U(i))^T \\ (\Phi_{i+N,i}(A_j)x(i) + B_{\phi_j}(i)U(i)) & S(i) \end{bmatrix} \;\ge\; 0 \tag{21}$$

for all L^N number of sets Ω_N in (13), and

$$\begin{bmatrix} S(i) & (A_jS(i) - B_jY(i))^T & (CS(i))^T & (R^{\frac{1}{2}}Y(i))^T \\ A_jS(i) - B_jY(i) & S(i) & 0 & 0 \\ CS(i) & 0 & \gamma_2(i) & 0 \\ R^{\frac{1}{2}}Y(i) & 0 & 0 & \gamma_2(i) \end{bmatrix} \;\ge\; 0 \tag{22}$$

for all $j \in [1, L]$, respectively, where

$$\begin{aligned} V_1(i) &= 2X_0^T(i)(I - \bar{A}_j(i))^{-T}\hat{Q}(I - \bar{A}_j(i))^{-1}\bar{B}_j(i), \quad X_0(i) = [x^T(i),\ 0,\ \cdots,\ 0]^T \\ V_0(i) &= X_0^T(i)(I - \bar{A}_j(i))^{-T}\hat{Q}(I - \bar{A}_j(i))^{-1}X_0(i) \\ V_2(i) &= \bar{B}_j^T(i)(I - \bar{A}_j(i))^{-T}\hat{Q}(I - \bar{A}_j(i))^{-1}\bar{B}_j(i) + \hat{R}. \end{aligned} \tag{23}$$

When $N = 1$, (20) is replaced by $\begin{bmatrix} \gamma_1(i) - x^T(i)Qx(i) & (R^{\frac{1}{2}}u(i))^T \\ R^{\frac{1}{2}}u(i) & I \end{bmatrix} \geq 0$. With the fixed $\gamma_2(i) = 1$, (9) can be replaced by

$$\begin{bmatrix} \gamma(i) - V_1(i)U(i) - V_0(i) & (V_2^{\frac{1}{2}}(i)U(i))^T & (\Phi_{i+N,i}(A_j)x(i) + B_{\phi_j}(i)U(i))^T \\ V_2^{\frac{1}{2}}(i)U(i) & I & 0 \\ (\Phi_{i+N,i}(A_j)x(i) + B_{\phi_j}(i)U(i)) & 0 & S(i) \end{bmatrix} \geq 0 \tag{24}$$

In the next subsection, we summarize how to implement the proposed RHC schemes and discuss some implementation issues.

3.1 Implementation of Robust RHC

Now, we can implement the proposed robust RHC schemes by using the following procedures.

a. At $i = 0$, increase N until (10) subject to (16), (19)-(22) is feasible (or (8) subject to (16), (19), (21), (22), and (24) with $\gamma_2(i) = 1$)
b. At each time i, solve problem (10) subject to (16), (19)-(22) (or (8) subject to (16), (19), (21), (22), and (24) with $\gamma_2(i) = 1$)
c. Among the solutions $u^*(\tau|i)$ for $\forall\ \tau \in [i, i + N - 1]$, implement the first control $u^*(i|i)$
d. At the next time $i + 1$, repeat the procedures (b) and (c).

As mentioned in Corollary 1, we can obtain another stabilizing robust RHC with the fixed $Q_f(0)$ and $\gamma_2(0)$ satisfying (16) and (18). We name the first RHC from (10) subject to (16), (19)-(22), the second RHC from (8) subject to (16), (19), (21), (22) and (24) with $\gamma_2(i) = 1$ and the third RHC from (8) subject to (19), (21), and (24) with the fixed $Q_f(0)$ and $\gamma_2(0)$ satisfying (16) and (18) as RHC1, RHC2, and RHC3, respectively.

RHC2 is conservative since γ_2 is fixed. As an approximation to RHC2, we consider (8) subject to (16), (19), (22) and (24) with $\gamma_2(i) = 1$ whose resulting control is called RHC4.

As mentioned in Remark 1, if we have many different pairs of (A_j, B_j), it is not easy to satisfy (16), (18), and (21). One way to overcome this problem is to use a different $H_j(i)$ for each j instead of $H(i)$ in (16) and (18). The RHC from (10) subject to (16), (19)-(22) with $H(i)$ replaced by $H_j(i)$ for each j is called RHC5, which is a slight modification of RHC1. Although RHC4 and RHC5 do not guarantee the cost monotonicity theoretically, they improve the feasibility and on-line computation time. Their performances are better than the previous results at least as shown in the following simulation examples.

Note that there always exists a cost horizon N satisfying $x_a^*(i+N|i) \in \mathcal{E}_{Q_f(i)}$ if the system is stabilizable, where the set of the feasible initial-state is given by

$$\xi_0 = \{x_0 \mid \exists U(0) \text{ such that } x_a^*(N) \in \mathcal{E}_{Q_f(0)}, \text{ where } Q_f(0) \text{ satisfies } (16) \text{ and } (22)\}.$$

4 Simulation Example

In this section we compare performances of the proposed RHCs with existing ones. For this purpose, we introduce a runtime cost of [9]:

$$J_{\text{run}} = \sum_{\tau=0}^{\text{runtime}} [x^T(\tau|\tau)Qx(\tau|\tau) + u^T(\tau|\tau)Ru(\tau|\tau)], \tag{25}$$

where runtime is set to 50. First, consider the following simulation example in [9]:

$$A(i) = \lambda(i)\begin{bmatrix}1 & 0.1\\ 0 & 1\end{bmatrix} + (1-\lambda(i))\begin{bmatrix}1 & 0.1\\ 0 & 1.2\end{bmatrix}, \quad B(i) = \begin{bmatrix}0\\ 0.0787\end{bmatrix}, \quad Q = C^TC, \quad C = \begin{bmatrix}1 & 0\end{bmatrix}$$

$R = 0.01$, $\lambda(i) = 0.5\sin(0.1i) + 0.5$, $u_{\max} = 2$, and $x_0 = [1\ \ 0]^T$.

Here, RHC1 and its approximation RHC5 have the largest feasible initial-state sets for the short horizon size, RHC2 is quite conservative whose relaxation is RHC 4, and RHC3 has the smallest computation time. As shown in this section, the proposed nonlinear feedback controls do not largely increase the computation time compared with the linear feedback controls such as in [1], while the proposed ones have better performances. The main reason for this phenomenon is that the proposed nonlinear controls make the closed-loop system go to the equilibrium faster and thus the constrained optimization problem becomes the unconstrained optimization problems quickly (then the proposed controls become linear feedback controls).

Figure 1 and Table 1 show performances and computation time of the proposed RHCs and existing ones, where RHC3 has the smallest computation time as expected. For feasibility, with $N = 1$, RHC2 is infeasible if $|x_1(0)| \geq 0.348$ and $x_2(0) = 0$, or $x_1(0) = 0$ and $|x_2(0)| \geq 0.524$. RHC2 with different variables $H_j(i)$ for each j is infeasible if $|x_1(0)| \geq 0.3677$ and $x_2(0) = 0$, or $x_1(0) = 0$ and $|x_2(0)| \geq 0.5243$. With $N = 1$, RHC1 and RHC3 are infeasible if $x_1(0) = 0$ and $|x_2(0)| \geq 0.79$ and if $x_1(0) = 0$ and $|x_2(0)| \geq 1.51$, respectively, while RHC4 and RHC5 make the state go to zero if $x_1(0) = 0$ and $|x_2(0)| \leq 1.7$ and if $x_1(0) = 0$ and $|x_2(0)| \leq 1.62$, respectively. This example illustrates that RHC4 and RHC5 have larger feasible initial-state sets, less computational burden, and better performance than RHC1 and RHC2. Thus, two approximate methods as well as RHC1 and RHC3 can be appropriate robust controllers for uncertain time-varying systems.

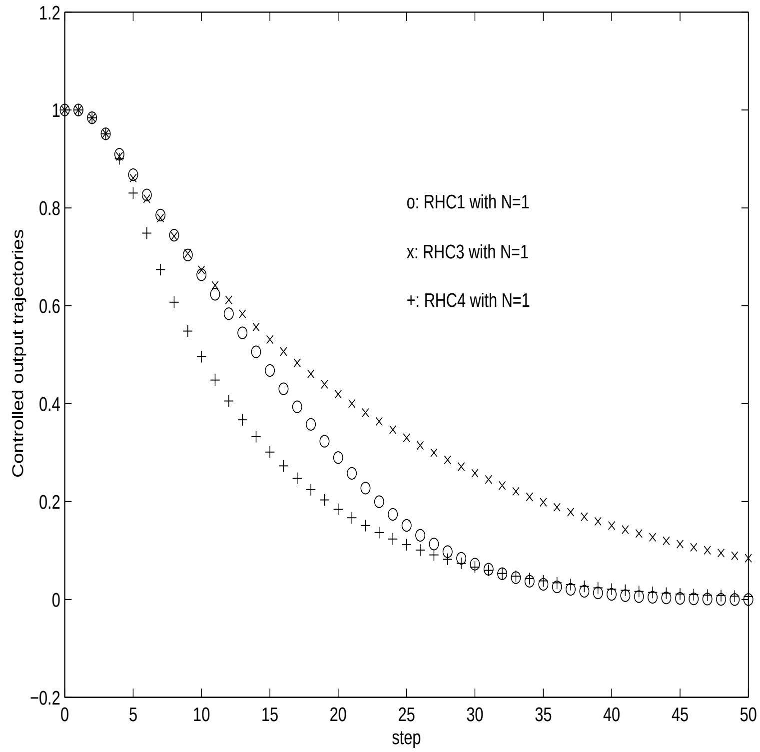

Figure 1: Controlled output: $y = Cx$

Table 1: Runtime costs J_{run} for the first example

Controller	J_{run}	Relative computation time
[1]	18.4	1
RHC1 with $N = 1$ ([8], [9])	10.8	3.5
[9]: best performance with $N = 5$	8.8	40
RHC1 with $N = 3$	9.0	7.8
RHC3 with $N = 1$	12.7	1.3
RHC3 with $N = 3$	9.4	4.0
RHC4 with $N = 1$	8.7	2.5
RHC4 with $N = 3$	8.4	5.9
RHC5 with $N = 1$	8.0	3.6
RHC5 with $N = 3$	7.9	6.8

Table 2: Runtime costs J_{run} for the second example

Controller	J_{run}	Relative computation time
[1]	41.5	1
RHC1 with $N = 1$ ([8], [9])	37.7	1.7
RHC1 with $N = 3$	32.2	6.6
RHC3 with $N = 1$	17.5	0.3
RHC3 with $N = 3$	18.6	2.6
RHC4 with $N = 1$	15.9	1.1
RHC4 with $N = 3$	16.4	6.7
RHC5 with $N = 1$	20.2	2.7
RHC5 with $N = 3$	33.6	6.2

Second, consider the following uncertain high-order system with three vertices:

$$\begin{aligned} A(i) &= \frac{1}{3}\left|\sin\left(\frac{i}{\alpha_1}\pi\right)\right| \begin{bmatrix} 0.83 & 0.083 & 0 & 0 \\ 0 & 0.83 & 0 & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{3}\left|\cos\left(\frac{i}{\alpha_2}\pi\right)\right| \begin{bmatrix} 0.83 & 0.083 & 0 & 0 \\ 0 & 0.996 & 0 & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1.2 \end{bmatrix} \\ &\quad + \alpha_3 \begin{bmatrix} 0.83 & 0.083 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B(i) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0787 \end{bmatrix}, \quad C = \begin{bmatrix} 0.83 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \end{aligned}$$

where $\alpha_1 = 5$, $\alpha_2 = 3$, $\alpha_3 = 1 - \frac{1}{3}\left|\sin(\frac{i}{\alpha_1}\pi)\right| - \frac{1}{3}\left|\cos(\frac{i}{\alpha_2}\pi)\right|$, $x_0 = [0.4 \;\; 0 \;\; 1 \;\; 0]^T$, runtime is set to 118, and the other parameters are the same as those of the first example.

Figure 2 illustrates that performances of the proposed control schemes are better than that in [1]. Table 2 shows that RHC3 with $N = 1$ and RHC4 with $N = 1$ have the smallest computation time and J_{run}, respectively. For this example, RHC1, RHC3, RHC4 and RHC5 have the smallest J_{run} when $N = 3$, $N = 1$, $N = 1$, and $N = 1$, respectively. For ease of comparison of feasibility, we have $x_2(0) = x_4(0) = 0$. Then, RHC1, RHC4, and RHC5 make the state go to zero for a very large initial value, i.e., they are almost feasible for any kind of initial value. The control method in [1] is infeasible if $|x_1(0)| \geq 23.2$ and $|x_3(0)| \geq 58$. With $N = 1$, RHC2 is infeasible if $|x_1(0)| \geq 0.0972$ and $|x_3(0)| \geq 0.243$. With $N = 1$, RHC3 is infeasible if $|x_1(0)| \geq 1.08$ and $|x_3(0)| \geq 2.7$. These performances are similar for many different values of α_1, α_2, and x_0, where Q, R, and u_{lim} hold.

5 Conclusion

In this paper, we proposed a new stabilizing receding horizon control (RHC) scheme for linear input-constrained discrete systems with polytopic uncertainty, which can easily be implemented by using

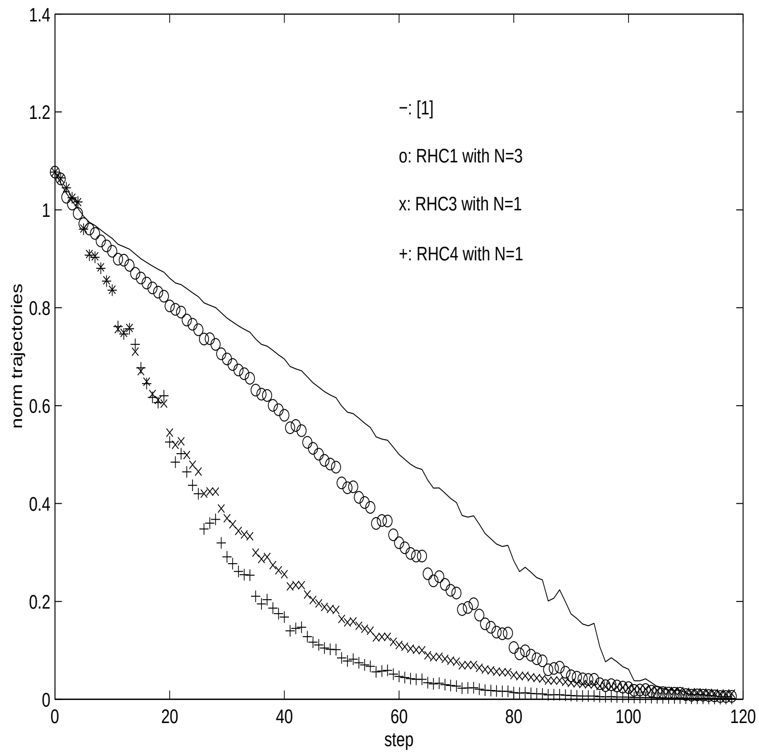

Figure 2: State norm ($\|x\|$) trajectories

linear matrix inequality (LMI) optimization. The control scheme is based on the minimization of the finite horizon cost with time-varying terminal weighting matrices. We had a rigorous proof of the closed-loop stability. We discussed modifications to the proposed scheme; for constrained uncertain systems, these modifications make the optimization problem more feasible numerically and the on-line computation time smaller than the original proposed scheme. Through simulation examples, we showed that the proposed schemes have a better performance, a less computational burden, or a wider feasible set than existing results in [1], [8], [9].

The proposed schemes in this paper are expected to be useful for various constrained robust control problems.

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A Proof of Lemma 1

The equation (5) comes from $\max_{j\in[1,L]} \Psi_j(i) = \sum_{k=1}^{L} \lambda_k(i) \max_{j\in[1,L]} \Psi_j(i) \geq \sum_{k=1}^{L} \lambda_k(i)\Psi_k(i)$ and

$$\begin{aligned} \sum_{j=1}^{L} \lambda_j(i)\Psi_j(i) - \Psi_b(i) &= \sum_{j=1}^{L} \lambda_j(i)\Psi_j(i) - \sum_{j=1}^{L} \lambda_j^2(i)\Psi_j(i) - 2\sum_{k=j+1}^{L}\sum_{j=1}^{L-1} \lambda_j(i)\lambda_k(i) \\ &\quad (A_j(i)x(i) + B_j(i)u(i))^T\Psi(i)(A_k(i)x(i) + B_k(i)u(i)) \\ &= \sum_{k=j+1}^{L}\sum_{j=1}^{L-1} \lambda_j(i)\lambda_k(i)A_{j,k}^T(i)\Psi(i)A_{j,k}(i) \\ &\geq 0 \text{ since } \Psi(i) \geq 0,\ \lambda_j(i) \geq 0,\ \lambda_k(i) \geq 0, \end{aligned} \tag{A.26}$$

where $A_{j,k}(i) = (A_j(i)x(i) + B_j(i)u(i)) - (A_k(i)x(i) + B_k(i)u(i))$.

The equation (6) comes from $A(i)x(i) + B(i)u(i) = \sum_{j=1}^{L}\lambda_j(i)(A_jx(i) + B_ju(i))$ and $\sum_{j=1}^{L}\lambda_j(i)$ $[G_1(A_jx(i)+B_ju(i))+G_2]^T\ \Psi(i)\ [G_1\ (A_jx(i)+B_ju(i))+G_2] \geq [G_1\sum_{j=1}^{L}\lambda_j(i)(A_jx(i)+B_ju(i)) + G_2]^T\ \Psi(i)\ [G_1\sum_{j=1}^{L}\lambda_j(i)\ (A_jx(i) + B_ju(i)) + G_2]$ by (A.26).

B Proof of Lemma 2

From Lemma 1, for any $\Psi(k) \geq 0$, $(A(i), B(i))$, $x(i)$, $u(i)$, $G_1(k)$, $G_2(k)$, and $T \geq 1$ we have

$$\max_{j\in[1,L]} \sum_{k=1}^{T}[G_1(k)(A_jx(i) + B_ju(i)) + G_2(k)]^T\Psi(k)[G_1(k)(A_jx(i) + B_ju(i)) + G_2(k)] \geq$$
$$\sum_{k=1}^{T}[G_1(k)(A(i)x(i) + B(i)u(i)) + G_2(k)]^T\Psi(k)[G_1(k)(A(i)x(i) + B(i)u(i)) + G_2(k)]. \tag{B.27}$$

From Lemma 1, $\max_{j\in[1,L]} \Psi_j(T|i) \geq \Psi_b(T|i)$ for any $(A(T|i), B(T|i))$. Let the pair $(A_j^*(T|i), B_j^*(T|i))$ be the solution for $\max_{j\in[1,L]} \Psi_j(T|i)$. Then

$$x^T(T|i)\Psi(T-1|i)x(T|i) + [G_1x(T|i) + G_2]^T\Psi(T|i)[G_1x(T|i) + G_2] \geq \sum_{\tau=T-1}^{T} \Psi_b(\tau|i), \tag{B.28}$$

where $x(T|i) = A(T-1|i)x(T-1|i) + B(T-1|i)u(T-1|i)$, $G_1 = A_j^*(T|i)$ and $G_2 = B_j^*(T|i)u(T|i)$. From (B.27), (B.28) is less than

$$\max_{j\in[1,L]}[x_j^T(T|i)\Psi(T-1|i)x_j(T|i) + (G_1x_j(T|i)+G_2)^T\Psi(T|i)(G_1x_j(T|i)+G_2)], \quad \text{(B.29)}$$

where $x_j(T|i) = A_j(T-1|i)x(T-1|i)+B_j(T-1|i)u(T-1|i)$. Let the pair $(A_j^*(T-1|i), B_j^*(T-1|i))$ be the solution of (B.29). Then

$$x^T(T-1|i)\Psi(T-2|i)x(T-1|i) + [G_1(1)x(T-1|i)+G_2(1)]^T\Psi(T-1|i)[G_1(1)x(T-1|i)+ G_2(1)] + [G_1(2)x(T-1|i)+G_2(2)]^T\Psi(T|i)[G_1(2)x(T-1|i)+G_2(2)] \geq \sum_{\tau=T-2}^{T}\Psi_b(\tau|i), \quad \text{(B.30)}$$

where $x(T-1|i) = A(T-2|i)x(T-2|i) + B(T-2|i)u(T-2|i)$, $G_1(1) = A_j^*(T-1|i)$, $G_2(1) = B_j^*(T-1|i)u(T-1|i)$, $G_1(2) = A_j^*(T|i)A_j^*(T-1|i)$, and $G_2(2) = A_j^*(T|i)B_j^*(T-1|i)u(T-1|i) + B_j^*(T|i)u(T|i)$. From (B.27), (B.30) is less than

$$\max_{j\in[1,L]}[x_j^T(T-1|i)\Psi(T-2|i)x_j(T-1|i) + (G_1(1)x_j(T-1|i)+G_2(1))^T\Psi(T-1|i)(G_1(1) x_j(T-1|i)+G_2(1)) + (G_1(2)x_j(T-1|i)+G_2(2))^T\Psi(T|i)(G_1(2)x_j(T-1|i)+G_2(2))], \quad \text{(B.31)}$$

where $x_j(T-1|i) = A_j(T-2|i)x(T-2|i) + B_j(T-2|i)u(T-2|i)$. Let the pair $(A_j^*(T-2|i), B_j^*(T-2|i))$ be the solution for (B.31). From the repeated procedure, we have the equation (7).

C Proof of Lemma 5

Let $\sigma = i+N$. Optimality shows that $J^*(i+1,\sigma+1)$ is less than that with $u^*(\cdot|i+1)$ and/or with $Q_f(i+1)$ replaced by any input-constrained $u(\cdot)$ and $Q_f(i)$. Similarly, $J^*(i,\sigma)$ is greater than that with $(A_j^*(\cdot|i), B_j^*(\cdot|i))$ replaced by any $(A_j(\cdot|i), B_j(\cdot|i))$.

Thus, replacing $u^*(\tau|i+1)$, $Q_f(i+1)$, and $(A_j^*(\tau|i), B_j^*(\tau|i))$ with $u^*(\tau|i)$, $Q_f(i)$, and $(A_j^*(\tau|i+1), B_j^*(\tau|i+1))$ for $\tau\in[i+1, i+N-1]$, respectively and $(A_j^*(i|i), B_j^*(i|i))$ with $(A(i), B(i))$ leads to

$$J^*(i+1,\sigma+1) - J^*(i,\sigma) \leq J(\sigma,\sigma+1) - x_b^T(\sigma)Q_f(i)x_b(\sigma) - [x^T(i)Qx(i) + u^{*T}(i)Ru^*(i)],$$

where $x_b(\sigma)$ is the state due to $x(i+1) = A(i)x(i)+B(i)u^*(i)$, $u(\tau|i)$, and $(A_j^*(\tau|i+1), B_j^*(\tau|i+1))$ for $\tau\in[i+1,i+N-1]$.

Since $x_a^*(\sigma)\in\mathcal{E}_{Q_f(i)}$, $u(\sigma) = -H(i)x_b(\sigma)$ satisfies (4) from Lemmas 3 and 4. With this $u(\sigma) = -H(i)x_b(\sigma)$, we have

$$\begin{aligned} J(\sigma,\sigma+1) - x_b^T(\sigma)Q_f(i)x_b(\sigma) &\leq x_b^T(\sigma)\{Q + H^T(i)RH(i) + \max_{j\in[1,L]}(A_j - B_jH(i))^T \\ &\quad Q_f(i)(A_j - B_jH(i)) - Q_f(i)\}x_b(\sigma) \\ &\leq 0 \ \text{by} \ (18). \end{aligned} \quad \text{(C.32)}$$

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